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A MIXED VARIATIONAL FORMULATION BASED ON EXACT INTRINSIC EQUATIONS FOR DYNAMICS OF MOVING BEAMS†

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Abstract—A nonlinear intrinsic formulation for the dynamics of initially curved and twisted beams in a moving frame is presented. The equations are written in a compact matrix form without any approximations to the geometry of the deformed beam reference line or to the orientation of the intrinsic cross-section frame. In accordance with previously published work, when effects of warping on the dynamics and local constraints on the cross-sectional deformation are ignorable, the in- and out-of-plane St Venant warping displacements (which are fully coupled for nonhomogeneous, anisotropic beams) need only be included explicitly in the determination of a suitable elastic law (a two-dimensional problem) and need not be considered explicitly in the one-dimensional equations governing global deformation. In this paper it is presupposed that such an elastic law is given as a one-dimensional strain energy function. Thus, the present equations, which are based on only six generalized strain variables, are valid for beams with closed cross-sections and for which warping is unrestrained. When simplified for various special cases, they agree with similar intrinsic equations in the literature. Although the resulting equations are Newtonian in structure (closely resembling Euler's dynamical equations for a rigid body), the formulation adheres to a variational approach throughout, thus providing a link between Newtonian and energy-based methods. In particular, the present development provides substantial insight into the relationships among variational formulations in which different displacement and rotational variables are used as well as between these formulations and Newtonian ones. For computational purposes, a compact and complete mixed variational formulation is presented that is ideally suited for development of finite element analyses. Finally, a specialized version of the intrinsic equations is developed in which shear deformations is suppressed.

INTRODUCTION

The modern theory of beams originated with the work of Kirchhoff and Clebsch as described by Love (1944). It is not the purpose of the present paper to review all the literature surrounding the development of beam theory. A recent paper on beam kinematics by Hodges (1987b) gives a brief survey in which the major works during the past few decades are categorized. The main purpose of this paper is to present a unifying framework in which other less general developments can be checked for consistency and, in light of which, differences that normally arise in technical theories can be interpreted.

In connection with specific technical theories of beams, disputes can arise because of differences in the way the displacement field is represented. Most commonly at the root of such controversies, one finds that different variables have been assumed by different investigators to describe finite rotation of the frame in which the material points in the cross-section experience only small displacements due to warping in and out of the cross-section plane. Another difference is the basis in which the displacements are measured. Such differences and some such disputes are discussed in Hodges *et al.* (1980) and Alkire (1984).

To facilitate the development of computational strategies for large deformation of beams, one may find that a mixed finite element formulation is preferable. A secondary purpose of this paper is to show how the present intrinsic formulation can be used to develop an elegant basis for mixed finite elements. The present formulation affords such a basis in that it is geometrically exact, it allows the use of very simple shape functions, and it assigns all force and momentum variables an easily identifiable physical meaning and definition. All these points have definite advantages in applications involving multiple-flexible-body systems (see Hodges *et al.*, 1989, for example).

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The present work builds strongly upon three sets of previously published works. First, the kinematical basis derived by Danielson and Hodges (1987, 1988) and Hodges (1987a,b) plays a major role in that the global kinematics of the deformed beam has been established completely in terms of Cartesian tensors and simple matrix notation. These concepts, respectively, obviate a lot of the complexity normally associated with curvilinear coordinate systems and allow for the compact expression of the equations.

Second, the recent work of Atilgan and Hodges (1989), which is based on asymptotic analyses such as Parker (1979), shows that for thin, moderately curved beams with closed cross-sections and without constraints on the warping, a linear two-dimensional cross-section deformation analysis can be performed to determine appropriate stiffnesses for use in the nonlinear one-dimensional global analysis. An example of such a two-dimensional analysis is given in Giavotto *et al.* (1983). This finding is taken as license in the present context to regard the elastic law as given in determining and solving the one-dimensional equations and, more important, to assert that the effects of in- and out-of-plane warping displacements (excluding end effects) are included implicitly in the structural equations. This was also the approach of Borri and Mantegazza (1985) and Borri and Merlini (1986).

Third, the intrinsic analyses presented in Reissner (1973, 1981) derive the strain measures from the internal forces and the virtual work. This work not only confirms that the one-dimensional strain measures derived by Danielson and Hodges (1987) from kinematical considerations alone are the appropriate ones, but it also shows that the intrinsic equations are *the* "God-given" equations for this problem from which all correct equations must be derivable. Similar strain measures are used by Simo and Vu-Quoc (1988) and by Iura and Atluri (1989).

It should be noted that with the present approach, it is possible to base the analysis on exact equilibrium and kinematical equations while regarding the constitutive law as approximate. The particular form of the constitutive law expressed herein implies that it is a function of the beam's initial twist and curvature as well as the material and geometric properties within a cross-section.

Before the actual development, some kinematical preliminaries are necessary, which are built upon the previous work of Danielson and Hodges (1987, 1988), Hodges (1987a,b) and Reissner (1973, 1981). The present derivation is kept strictly within the confines of a variational approach, starting with Hamilton's principle. After the equations are obtained, they are compared with those of previous authors. A specialized version of the equations for beams with zero shear deformation is also derived and compared with previous work. Finally, a mixed variational formulation is presented on which powerful finite element solution procedures can be built. This approach is believed to be simpler than a displacement analysis of comparable capability (see Cardona and Geradin, 1988, for example).

KINEMATICAL PRELIMINARIES

In this section a notation convention for vector-dyadic quantities is presented, the beam configuration is described, appropriate base vectors are introduced, and the strain measures and velocity field are developed. Although the present formulation of equilibrium equations is intrinsic (and thus, not specific for any particular set of displacement variables), it is expedient to temporarily introduce the displacement, perform certain operations with it, and then eliminate it to obtain an intrinsic formulation.

Finite rotation and vector-dyadic notation

For the purpose of describing the operations below, let us introduce two arbitrary frames b and B in which are fixed dextral triads $\mathbf{b}_i$ and $\mathbf{B}_i$ for $i = 1, 2, 3$. One can denote rotation from b to B by pre-dot multiplication with an orthogonal tensor which is called the global rotation tensor $\mathbf{C}$ so that

$$\mathbf{B}_i = \mathbf{C} \cdot \mathbf{b}_i = C_{ij}\mathbf{b}_j \tag{1}$$

where repeated indices are summed over their range. It should be noted that by the present convention, the matrix C is the transpose of the matrix used by Kane *et al.* (1983) and that C is the transpose of the usual matrix of components of the Cartesian tensor **C**. The global rotation tensor can be represented as a linear combination of the dyads formed from the base vectors. In particular,

$$\mathbf{C} = \mathbf{B}_i \mathbf{b}_i. \tag{2}$$

Rotation from $\mathbf{B}_i$ to $\mathbf{b}_i$ is accomplished by pre-dot multiplication with the transpose of **C**. Thus,

$$\mathbf{b}_i = \mathbf{C}^{\mathrm{T}} \cdot \mathbf{B}_i. \tag{3}$$

The components C_{ij} are the direction cosines

$$C_{ij} = \mathbf{B}_i \cdot \mathbf{b}_j. \tag{4}$$

The rotation tensor **C** and corresponding matrix C could be written with superscripts (i.e. $\mathbf{C}^{Bb}$) to emphasize that the above relations apply to these two frames. The transpose could then be written by reversing the superscripts so that $\mathbf{C}^{bB} = (\mathbf{C}^{Bb})^{\mathrm{T}}$. Because the frames B and b are so commonly used, these superscripts are omitted. However, analogous relations apply to the orientation of any frame relative to another in the same way these relate to the orientation of B in b. In those relatively rare occasions when frames other than B and b are needed, the superscripted form will be used; for example, the orientation of B in another frame A will be written as $\mathbf{C}^{BA}$ or C^{BA}. Note that

$$\mathbf{C}^{BA} = \mathbf{C}^{Bb} \cdot \mathbf{C}^{bA}. \tag{5}$$

Similar relations apply to matrices

$$C^{BA} = C^{Bb} C^{bA}. \tag{6}$$

Although the fundamental relations for the foregoing theory are in Cartesian tensor (i.e. dyadic) notation, most of the development will proceed with standard matrix operations. For this reason, the relation between the vector-dyadic form and the matrix form needs to be addressed before going further. An arbitrary vector **Z** can be expressed in terms of any bases whatsoever. Thus,

$$\begin{aligned} \mathbf{Z} &= Z_{Ai} \mathbf{A}_i \\ &= Z_{bi} \mathbf{b}_i \\ &= Z_{Bi} \mathbf{B}_i \end{aligned} \tag{7}$$

where $\mathbf{A}_i$, for $i = 1, 2, 3$, is a dextral triad fixed in A. It should be obvious that the first subscript of Z is indicative of the basis in which the measure numbers are defined. A compact matrix notation is adopted for the present work in which the measure numbers of Z are arranged in a column matrix

$$Z_b = \begin{Bmatrix} Z_{b1} \\ Z_{b2} \\ Z_{b3} \end{Bmatrix}. \tag{8}$$

It is also necessary at times to refer to the corresponding dual matrix which has the same measure numbers but arranged antisymmetrically

$$\tilde{Z}_b = \begin{bmatrix} 0 & -Z_{b3} & Z_{b2} \\ Z_{b3} & 0 & -Z_{b1} \\ -Z_{b2} & Z_{b1} & 0 \end{bmatrix}. \tag{9}$$

The ($\tilde{\ }$) operator (sometimes called a "cross product operator" for obvious reasons) also applies to vectors. The *tensor* $\tilde{\mathbf{Z}}$ has components in the b basis given by the matrix $\tilde{Z}_b$ such that

$$\begin{aligned} \tilde{\mathbf{Z}} &= \mathbf{Z} \times \boldsymbol{\Delta} \\ &= \mathbf{b}_i \tilde{Z}_{bij} \mathbf{b}_j \end{aligned} \tag{10}$$

where $\boldsymbol{\Delta}$ is the identity dyadic. (Note that this convention for naming the matrix of components for a tensor does not apply to the finite rotation tensor and its corresponding matrix of direction cosines.)

With this notation it is wise to be mindful of certain identities regarding the ($\tilde{\ }$) operator. When Y and Z are 3×1 column matrices, it is easily shown that

$$\begin{aligned} (\tilde{Z})^{\mathrm{T}} &= -\tilde{Z} \\ \tilde{Z}Z &= 0 \\ \tilde{Y}Z &= -\tilde{Z}Y \\ Y^{\mathrm{T}}\tilde{Z} &= -Z^{\mathrm{T}}\tilde{Y} \\ \tilde{Y}\tilde{Z} &= ZY^{\mathrm{T}} - \Delta Y^{\mathrm{T}}Z \\ \widetilde{\tilde{Y}Z} &= \tilde{Z}\tilde{Y} + \tilde{Y}\tilde{Z} \end{aligned} \tag{11}$$

where Δ is the 3×3 identity matrix.

Beam configuration and base vectors

Let x_1 denote length along a curved reference line r within an undeformed, but initially curved and twisted beam; a schematic is shown in Fig. 1. A particle on the beam reference

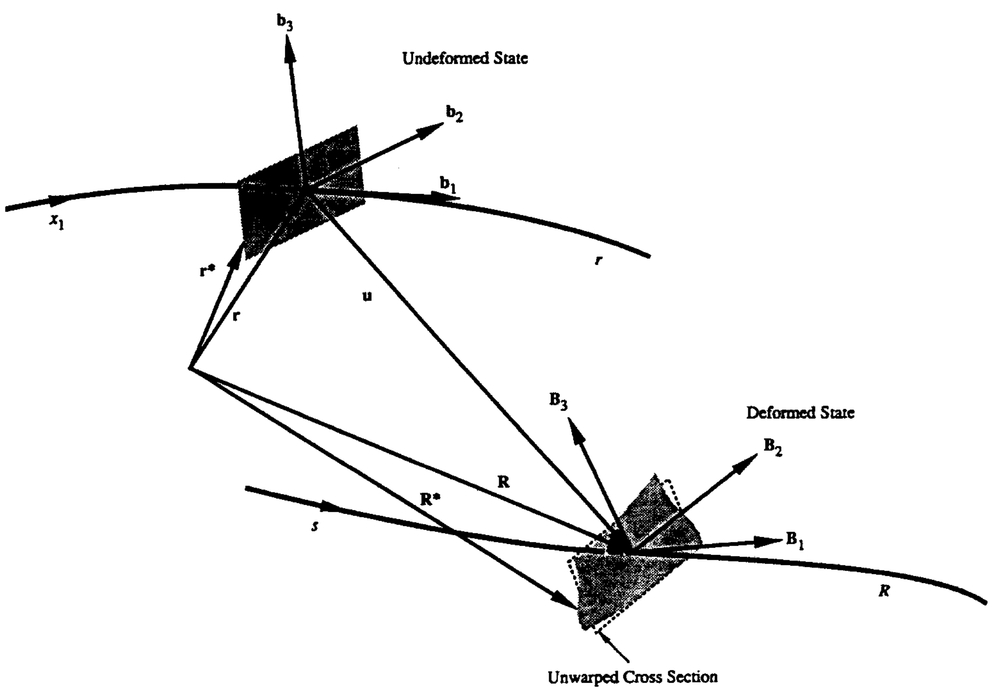

Fig. 1. Schematic of undeformed and deformed beam reference lines and reference cross-section; note that $\mathbf{B}_1 = \mathbf{B}_2 \times \mathbf{B}_3$ and that the unwarped cross-section (shown dashed) has undergone simple rigid body translation and rotation.

line is located relative to a point fixed in a reference frame A by the position vector $\mathbf{r}(x_1)$. A is an absolute frame as far as deformation is concerned in that the orientation of the local undeformed beam cross-section in A is a function only of x_1 and not of time t. The motion of A in an inertial frame I is, however, supposed to be known for all time. Let x_2 and x_3 denote lengths along lines orthogonal to the reference line $\boldsymbol{r}$ within a cross-section $A(x_1)$. At each point along $\boldsymbol{r}$ define a frame b in which are fixed orthogonal unit vectors $\mathbf{b}_i$ for $i = 1, 2, 3$ such that $\mathbf{b}_2(x_1)$ and $\mathbf{b}_3(x_1)$ are tangent to the coordinate curves x_2 and x_3 at $\boldsymbol{r}$ and $\mathbf{b}_1$ is tangent to $\boldsymbol{r}$. The frame b has an orientation which is fixed in A for any fixed value of x_1 but varies along the beam if the beam is initially curved or twisted. Thus, an arbitrary point in the undeformed beam can be located by

$$\mathbf{r}^* = \mathbf{r} + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 = \mathbf{r} + \boldsymbol{\xi}. \tag{12}$$

Each value of x_1 specifies not only a point on $\boldsymbol{r}$ but also a reference cross-section at that point. Notice that $\boldsymbol{\xi}$ is the position vector of an arbitrary point within a particular cross-section relative to the point in that cross-section where $\boldsymbol{r}$ intersects it.

Similarly, consider the configuration of the deformed beam as shown in Fig. 1. The locus of material points along $\mathbf{r}$ has now assumed a different curved line denoted by $\boldsymbol{R}$. Let s denote length along $\boldsymbol{R}$. At each point along $\boldsymbol{R}$ introduce a frame B in which are fixed orthogonal unit vectors $\mathbf{B}_i(x_1)$ for $i = 1, 2, 3$. The meaning of this "intrinsic" frame is discussed in Danielson and Hodges (1987, 1988) and Simo (1985). Note that $\mathbf{B}_1 = \mathbf{B}_2 \times \mathbf{B}_3$ is not necessarily tangent to $\boldsymbol{R}$ unless the Euler–Bernoulli hypothesis (that the reference cross-section remains normal to $\boldsymbol{R}$ when the beam is deformed) is adopted. Introducing $\mathbf{R} = \mathbf{r} + \mathbf{u}$ where $\mathbf{u}$ is the displacement of a particle on the reference line, one can represent the position of a particle in the deformed beam which had position $\mathbf{r}^*$ in the undeformed beam as

$$\begin{aligned} \mathbf{R}^* &= \mathbf{R} + \mathbf{C}\cdot(\boldsymbol{\xi} + \mathbf{w}) \\ &= \mathbf{r} + \mathbf{u} + x_2\boldsymbol{B}_2 + x_3\mathbf{B}_3 + w_i\mathbf{B}_i \end{aligned} \tag{13}$$

where $\mathbf{w} = w_i\mathbf{b}_i$ is a vector which represents the (small) warping displacement field such that $w_i = w_i(x_1, x_2, x_3)$. Except for x_2, x_3 and w_i, all quantities in this equation depend only on x_1. The pre-dot-multiplication by $\mathbf{C}$ (a so-called push-forward operation) has the effect of embedding the vector $\boldsymbol{\xi} + \mathbf{w}$ in some intermediate frame β, which has base vectors that coincide with those of b, and then rotating β so that its base vectors finally coincide with those of B. Notice that if warping is suppressed, the locus of points that were in a cross-section plane in the undeformed beam are now in the plane determined by $\mathbf{B}_2$ and $\mathbf{B}_3$. It should be emphasized that the unit vectors $\mathbf{B}_i$, for $i = 1, 2, 3$, are orthogonal by definition and that the material lines in the deformed beam that were along $\mathbf{b}_i$ in the undeformed beam are not necessarily orthogonal in the deformed beam. Although two displacement measures ($\mathbf{u}$ and $\mathbf{w}$) have been introduced, they subsequently will be eliminated.

Generalized strain measures

The Jaumann–Biot–Cauchy (engineering) strain for a beam undergoing arbitrary deformation was derived by Danielson and Hodges (1987). In that work, the three-dimensional strain field in the beam is developed in terms of one-dimensional generalized strains whose definitions agree with those from the Cosserat theory (a director theory). The vector-dyadic definitions of the generalized strains can be expressed as

$$\begin{aligned} \boldsymbol{\gamma} &= \mathbf{C}^{\mathrm{T}}\cdot\mathbf{R}' - \mathbf{r}' \\ \boldsymbol{\kappa} &= \mathbf{C}^{\mathrm{T}}\cdot\mathbf{K} - \mathbf{k} \end{aligned} \tag{14}$$

where $(\ )'$ denotes the derivative with respect to x_1 (for vectors, the derivative is taken in A) and $\mathbf{K}$ is the curvature vector of the deformed beam as defined by Danielson and Hodges (1987) such that

$$\tilde{\mathbf{K}} = (\mathbf{C}^{BA})' \cdot \mathbf{C}^{AB}. \tag{15}$$

For small strain when **K** is expressed in the B basis, K_{B1} is the twist per unit length and K_{B2} and K_{B3} are curvatures of the deformed beam. Similarly, **k** is the curvature vector for the undeformed beam

$$\tilde{\mathbf{k}} = (\mathbf{C}^{bA})' \cdot \mathbf{C}^{Ab} \tag{16}$$

and when **k** is expressed in the b basis, k_{b1} is the twist per unit length and k_{b2} and k_{b3} are curvatures of the undeformed beam.

The vector-dyadic form of the generalized strains as in eqn (14) can be especially illuminating. For the "force strain" (so-called because its components are conjugate to the section forces) $\boldsymbol{\gamma}$, the tangent vector (multiplied by the stretch s') $\mathbf{R}'$ of the deformed beam is "pulled back" to the undeformed orientation by the premultiplication with $\mathbf{C}^{\mathrm{T}}$; then the strain is determined by subtracting the tangent vector of the undeformed beam $\mathbf{r}'$. For the "moment strain" (so-called because its components are conjugate to the section moments) $\boldsymbol{\kappa}$, the curvature vector (multiplied by the stretch s') **K** of the deformed beam is pulled back to the undeformed orientation by the premultiplication with $\mathbf{C}^{\mathrm{T}}$; then the strain is determined by subtracting the curvature vector of the undeformed beam **k**. In both cases, the global rotation is "removed" for the deformation in accordance with the polar decomposition theorem as applied by Danielson and Hodges (1987).

These generalized strains can be expressed in terms of two column matrices whose elements are the measure numbers of the force and moment strain vectors in the b basis. If the force strain is denoted by

$$\gamma = \begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \end{Bmatrix} \tag{17}$$

then the strain–displacement relation from the first of eqns (14) can be expressed in several ways:

$$\begin{aligned} \gamma &= (r_B + u_B)' + \tilde{K}_B(r_B + u_B) - C^{\mathrm{T}}(r_B' + \tilde{K}_B r_B) \\ &= C[(r_b + u_b)' + \tilde{k}_b(r_b + u_b)] - (r_b' + \tilde{k}_b r_b) \\ &= C^{BA}(r_A + u_A)' - C^{bA} r_A'. \end{aligned} \tag{18}$$

For an initially straight beam, if $\mathbf{b}_1 = \mathbf{A}_1$ is parallel to the undeformed beam axis, the third of eqns (18) reduces to the strain used in Hodges (1987a). If the beam is further supposed to be rigid in shear, it reduces to that of Hodges (1985). The fact that $\mathbf{b}_1$ is tangent to $\boldsymbol{r}$ allows the simplification that $r_b' + \tilde{k}_b r_b = e_1$ where

$$e_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \tag{19}$$

so that the simplest form of γ becomes

$$\gamma = C(e_1 + u_b' + \tilde{k}_b u_b) - e_1 \tag{20}$$

where u_b is the column matrix containing the measure numbers of **u** in the b basis.

Similarly, if the moment strain is denoted by

$$\kappa = \begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} \tag{21}$$

then the strain–displacement relation can be obtained from the second of eqns (14) along with eqns (16) and (15) as

$$\kappa = K_B - k_b \tag{22}$$

where

$$\tilde{K}_B = -C'C^T + C\tilde{k}_b C^T. \tag{23}$$

In both the force strain and the moment strain, the measure numbers of the initial curvature vector, contained in k_b, are known.

INTRINSIC EQUATIONS FROM HAMILTON'S PRINCIPLE

It is believed that the development of mixed finite elements for beams is facilitated by keeping the intrinsic theory separate from the kinematical relations so that each of the primitive relations needed are in the simplest possible form. The intent of the present section is to provide the intrinsic formulation. Therefore, although the displacements **u** and **w** have been introduced, they will be subsequently eliminated in order to obtain a truly intrinsic theory. It will be necessary to include **u**, however, when applying the theory such as in the mixed form of Hamilton's weak principle (see Borri *et al.*, 1985) presented in a later section.

In this section the intrinsic equations are derived from Hamilton's principle which can be written as

$$\int_{t_1}^{t_2} \int_0^l [\delta(K-U) + \overline{\delta W}]\, dx_1\, dt = \overline{\delta \mathscr{A}} \tag{24}$$

where t_1 and t_2 are arbitrary fixed times, K and U are the kinetic and strain energy densities per unit length, respectively, $\overline{\delta \mathscr{A}}$ is the virtual action at the ends of the beam and at the ends of the time interval (see Peters and Izadpanah, 1988, for discussion of this term and its importance), and $\overline{\delta W}$ is the virtual work of applied loads per unit length. The bars are used to indicate that the virtual work and the virtual action need not be the variations of functionals.

Internal forces from strain energy

The warping displacements w_i, a three-dimensional displacement field superimposed on the motion of the frame B, is small. Since the warping displacements are small, the influence of warping may be ignored in the inertial and applied load terms. However, the gradients of the warping displacement field are not necessarily small, and thus, the strain field depends upon the warping. In the present development, following the work of Borri and Merlini (1986), the contribution of warping to the strain will be according to a Saint-Venant solution in terms of stress resultants for a given beam cross-section. A good approximation for the warping can be obtained from a finite element approach such as that of Giavotto *et al.* (1983). This way, the warping influences only the elastic constants in a beam constitutive law that is written in terms of generalized strains γ and κ. This implies that the strain energy per unit length can be symbolically expressed as

$$U = U(\gamma, \kappa). \tag{25}$$

In the present work, it is assumed that such a strain energy density is known. It should be noted that the strain energy function U depends on the beam geometric parameters such as initial twist, initial curvature, and taper, and on cross-section materials and geometry.

In Danielson and Hodges (1987) the strain field in the beam is shown to be a nonlinear function of γ and κ (which are in turn nonlinear functions of some set of displacement and orientation variables). For any arbitrary set of displacement and orientation variables, a geometric stiffness effect will arise naturally in the intrinsic equations because of the definitions of γ and κ. For example, the increased effective "bending" stiffness due to axial force stems from this type of geometric stiffness. However, any geometric stiffness arising from the nonlinearity between the strain field and γ and κ must be handled in terms of a separate calculation. Borri and Mantegazza (1985) and Borri and Merlini (1986) have shown approximate ways to obtain this type of geometric stiffness, which is important for beams with open cross-sections. For example, the so-called trapeze effect arises from this type of geometric stiffness. Within the framework of Danielson and Hodges (1987), an analogous calculation could be carried out, but determining this aspect of the constitutive law will be deferred to a later paper.

The influence of restrained warping due to end conditions can be treated with additional kinematical variables in a manner similar to Danielson and Hodges (1987). The addition of such variables, while straightforward in the context of a beam analysis, is also not treated in the present paper. For the analysis of beams with closed cross-sections, the present treatment is adequate in most cases. For a discussion of exceptions see Rehfield *et al.* (1988).

Now given the strain energy in the form of eqn (25), one can carry out the variation required in eqn (24) so that

$$\int_0^l \delta U \, dx_1 = \int_0^l \left[\delta\gamma^T \left(\frac{\partial U}{\partial \gamma}\right)^T + \delta\kappa^T \left(\frac{\partial U}{\partial \kappa}\right)^T \right] dx_1. \tag{26}$$

Following Danielson and Hodges (1988), the partial derivatives of U are identified as section stress resultants

$$F_B = \left(\frac{\partial U}{\partial \gamma}\right)^T$$
$$M_B = \left(\frac{\partial U}{\partial \kappa}\right)^T \tag{27}$$

where F_B and M_B are column matrices which contain measure numbers of internal forces and moments, respectively. The first element of F_B is the axial force and the second and third elements are shear forces, all expressed in the deformed beam basis B. Similarly, the first element of M_B is the twisting moment and the second and third elements are bending moments, again in the deformed beam basis B.

In order to obtain the the contribution of the internal forces to the equations of motion, it is necessary to express variations of the strain measures (i.e. $\delta\gamma$ and $\delta\kappa$) in terms of quantities that are independent of the variables used to express the displacement and rotation of the beam reference line and cross-section, respectively. First, the variation of κ is obtained. Making use of eqn (23) and the fact that $\delta\kappa = \delta K_B$, one obtains

$$\delta\tilde{\kappa} = -\delta C' C^T - C' \delta C^T + \delta C \tilde{k}_b C^T + C \tilde{k}_b \delta C^T. \tag{28}$$

A rigorous treatment of virtual rotation in dynamics may be found in Kane (1968). For beam problems, it is usually treated in terms of the Kirchhoff kinetic analogy as discussed in Love (1944); see, for example, Reissner (1973), Hodges *et al.* (1980) and Hodges (1985). In accordance with these references, the virtual rotation can be found by replacing ( )′ with δ( ) in eqn (23) and ignoring other terms so that

$$\widetilde{\overline{\delta\psi}}_B = -\delta C C^{\mathrm{T}}. \tag{29}$$

In this expression of virtual rotation the overbar indicates that $\overline{\delta\psi}_B$ is not the variation of a function, as can be observed. Differentiation of eqn (29) with respect to x_1 leads to

$$\widetilde{\overline{\delta\psi}}'_B = \delta C' C^{\mathrm{T}} - \delta C C'^{\mathrm{T}} \tag{30}$$

which can be used to eliminate the first term on the right-hand side of eqn (28). One can then use eqns (22) and (23) to eliminate C' so that

$$C' = C\tilde{k}_b - \tilde{K}_B C \tag{31}$$

and eqn (29) to eliminate δC so that

$$\delta C = -\widetilde{\overline{\delta\psi}}_B C. \tag{32}$$

Substitution of eqns (30)–(32) then yields

$$\delta\tilde{\kappa} = \widetilde{\overline{\delta\psi}}'_B + \tilde{K}_B \widetilde{\overline{\delta\psi}}_B - \widetilde{\overline{\delta\psi}}_B \tilde{K}_B. \tag{33}$$

Using the sixth of eqns (11), one can then obtain a simple relation for the variation of the moment strain

$$\delta\kappa = \overline{\delta\psi}'_B + \tilde{K}_B \overline{\delta\psi}_B. \tag{34}$$

This equation is sometimes referred to as a transpositional relation and is analogous to that rederived below for angular velocity. If one regards κ as the derivative with respect to x_1 of a set of spatial quasi-coordinates, it shows that the derivative with respect to x_1 and the variation are not commutative when applied to these quasi-coordinates.

The variation of the force strain can be carried out in a similar fashion. From eqn (20) the variation is

$$\delta\gamma = \delta C(e_1 + u'_b + \tilde{k}_b u_b) + C(\delta u'_b + \tilde{k}_b \delta u_b). \tag{35}$$

The first term on the right-hand side can be written as

$$\delta C(e_1 + u'_b + \tilde{k}_b u_b) = \delta C C^{\mathrm{T}}(e_1 + \gamma). \tag{36}$$

For the second term, one can introduce a virtual displacement vector $\overline{\delta\mathbf{q}} = {}^b\delta(\mathbf{u}) = {}^A\delta(\mathbf{u})$ such that $\overline{\delta q}_b = \delta u_b$, and thus eliminate u_b completely. Although when expressed in the b basis the virtual displacements are variations of the displacement, this is not true in any other bases. Thus, an overbar is used here as with the virtual rotation with the exception that, here, the bar can be dropped in the case when the virtual displacement is expressed in the b basis whereas with virtual rotation the bar can never be dropped. Differentiating the expression

$$C\delta u_b = C\overline{\delta q}_b = \overline{\delta q}_B \tag{37}$$

one can obtain an expression for $C\delta u'_b$ that is independent of u_b

$$C\delta u'_b = \overline{\delta q}'_B - C' C^{\mathrm{T}} \overline{\delta q}_B. \tag{38}$$

From eqns (31) and (35)–(38) now one can write

$$\delta\gamma = \overline{\delta q}'_B + \tilde{K}_B \overline{\delta q}_B + (\tilde{e}_1 + \tilde{\gamma})\overline{\delta\psi}_B. \tag{39}$$

Equation (34) and (39) are the relations needed to express the variation of the strain energy in terms of intrinsic quantities only. These equations agree with similar relations derived by Reissner (1973) in a completely different manner. Now the variation of the strain energy density can be written as

$$\begin{aligned}\delta U &= \delta\gamma^{\mathrm{T}} F_B + \delta\kappa^{\mathrm{T}} M_B \\ &= [(\overline{\delta q}'_B)^{\mathrm{T}} - \overline{\delta q}^{\mathrm{T}}_B \tilde{K}_B - \overline{\delta\psi}^{\mathrm{T}}_B (\tilde{e}_1 + \tilde{\gamma})] F_B + [(\overline{\delta\psi}'_B)^{\mathrm{T}} - \overline{\delta\psi}^{\mathrm{T}}_B \tilde{K}_B] M_B.\end{aligned} \tag{40}$$

It is interesting to note that the variation of the strain energy density is not only independent of any displacement or orientation variable, but that it is also material independent.

Inertial forces from kinetic energy

The inertial forces for a beam that is moving in an accelerating, rotating frame A can be derived from the kinetic energy

$$\int_0^l K \,\mathrm{d}x_1 = \frac{1}{2}\int_0^l \iint_{A(x_1)} \rho \mathbf{v}^{MI} \cdot \mathbf{v}^{MI} \sqrt{g} \,\mathrm{d}x_2 \,\mathrm{d}x_3 \,\mathrm{d}x_1 \tag{41}$$

where

$$\sqrt{g} = 1 - x_2 k_{b3} + x_3 k_{b2} > 0 \tag{42}$$

and ρ is the mass density, M is an arbitrary material point, and $\mathbf{v}^{MI}$ is the velocity of M in I. Application of elementary laws of kinematics (such as found in Kane and Levinson, 1985) allows one to write

$$\mathbf{v}^{MI} = \mathbf{v}^{b^*I} + \boldsymbol{\omega}^{bI} \times \mathbf{p}^{B^*/b^*} + {}^b\dot{\mathbf{p}}^{B^*/b^*} + \boldsymbol{\omega}^{BI} \times \mathbf{p}^{M/B^*} + {}^B\dot{\mathbf{p}}^{M/B^*} \tag{43}$$

where

$$\boldsymbol{\omega}^{BI} = \boldsymbol{\omega}^{Bb} + \boldsymbol{\omega}^{bI}. \tag{44}$$

Here ${}^B(\dot{\ })$ represents the time derivative in B, ${}^b(\dot{\ })$ represents the time derivative in b, $\mathbf{p}$ represents a generic position vector, $\boldsymbol{\omega}$ a generic angular velocity vector, and $\mathbf{v}$ a generic velocity vector. The right superscripts indicate the points and/or frames involved. The point b^* is a point on the reference axis of the beam before deformation; the point B^* is the same material point in the deformed beam. These vectors can be identified with the configuration of the beam rather easily. The part of $\mathbf{p}^{M/B^*}$ which is due to warping will be ignored in the inertial forces development. The displacement vector $\mathbf{u} = \mathbf{p}^{B^*/b^*}$. Since the motion of b is known in I, other vectors can be associated with that known motion. The velocity of b^* in I is $\mathbf{v}^{b^*I}$; its measure numbers in b are known and the column matrix containing those measure numbers is denoted by v_b. Similarly, the inertial angular velocity of the frame b is $\boldsymbol{\omega}^{bI}$; thus its measure numbers in b are also known and the column matrix containing those measure numbers is denoted by ω_b.

Now with the introduction of suitable generalized speeds in accordance with Kane and Levinson (1985), it is possible to eliminate the displacements from this part of the formulation as well. First introduce the column matrix which contains the measure numbers of $\boldsymbol{\omega}^{BI}$ in the B basis denoted by $\boldsymbol{\Omega}_B$. The definition of angular velocity as given by Kane and Levinson (1985) reveals that

$$\tilde{\Omega}_B = -\dot{C}C^T + C\tilde{\omega}_b C^T \tag{45}$$

where $(\dot{\ })$ is the derivative with respect to time. The similarity of this equation with eqn (23) is duly noted.

Now one can introduce the column matrix which contains the measure numbers of $\mathbf{v}^{B^*I}$ in the B basis denoted by V_B. Equation (43) reveals that

$$V_B = C(v_b + \dot{u}_b + \tilde{\omega}_b u_b). \tag{46}$$

The similarity between this relation and the strain in eqn (20) is also apparent. With these generalized speeds, the inertial velocity of M expressed in the B basis is

$$v_B^{MI} = V_B + \tilde{\Omega}_B \xi_B \tag{47}$$

where

$$\xi_B = p_B^{M/B^*} = \begin{Bmatrix} 0 \\ x_2 \\ x_3 \end{Bmatrix}. \tag{48}$$

Now the kinetic energy per unit length K can be written in terms of the generalized speeds in a very compact notation. Introducing the mass per unit length, and the first and second distributed mass moments of inertia

$$m = \iint_{A(x_1)} \rho\sqrt{g}\, dx_2\, dx_3$$

$$m\bar{\xi}_B = \iint_{A(x_1)} \rho\xi_B\sqrt{g}\, dx_2\, dx_3$$

$$i_B = \iint_{A(x_1)} \rho(\xi_B^T\xi_B\Delta - \xi_B\xi_B^T)\sqrt{g}\, dx_2\, dx_3 \tag{49}$$

one can simply write

$$K = \tfrac{1}{2}(mV_B^T V_B - 2m\Omega_B^T \tilde{V}_B \bar{\xi}_B + \Omega_B^T i_B \Omega_B). \tag{50}$$

Now, with the use of the definitions of the generalized speeds, eqns (45) and (46), one can proceed as was done above with the strain energy to obtain expressions for the variations of V_B and Ω_B that are independent of displacement variables. First, the variation of Ω_B is obtained. Making use of eqn (45) one obtains

$$\delta\tilde{\Omega}_B = -\delta\dot{C}C^T - \dot{C}\delta C^T + \delta C\tilde{\omega}_b C^T + C\tilde{\omega}_b \delta C^T. \tag{51}$$

Differentiation of eqn (29) with respect to t leads to an expression in terms of δC, $\dot{C}$, and $\delta\dot{C}$. By using this, eqn (45), and eqn (51), one can obtain

$$\delta\tilde{\Omega}_B = \dot{\widetilde{\delta\psi}}_B + \tilde{\Omega}_B\widetilde{\delta\psi}_B - \widetilde{\delta\psi}_B\tilde{\Omega}_B. \tag{52}$$

From the sixth of eqn (11) a simple relation emerges

$$\delta\Omega_B = \dot{\overline{\delta\psi}}_B + \tilde{\Omega}_B\overline{\delta\psi}_B. \tag{53}$$

This equation provides a bridge between variational and Newtonian methods in that intrinsic equations are usually Newtonian, and this relation allows one to develop intrinsic equations from Hamilton's principle. If one regards Ω_B as the derivative with respect to t of a set of temporal quasi-coordinates, it shows that the time derivative and the variation are not commutative when applied to the quasi-coordinates.

The variation of V_B can be carried out in a similar fashion to the way it was done for Ω_B. Taking the variation of eqn (46) one obtains

$$\delta V_B = \delta C(v_b + \dot{u}_b + \tilde{\omega}_b u_b) + C(\delta\dot{u}_b + \tilde{\omega}_b \delta u_b). \tag{54}$$

The first term can be written as

$$\delta C(v_b + \dot{u}_b + \tilde{\omega}_b u_b) = \delta C C^{\mathrm{T}} V_B. \tag{55}$$

For the second term, one can differentiate eqn (37) with respect to t and obtain an expression for $C\delta\dot{u}_b$

$$C\delta\dot{u}_b = \dot{\overline{\delta q}}_B - \dot{C}C^{\mathrm{T}}\overline{\delta q}_B. \tag{56}$$

From eqns (45) and (54)–(56) one can now write

$$\delta V_B = \dot{\overline{\delta q}}_B + \tilde{\Omega}_B\overline{\delta q}_B + \tilde{V}_B\overline{\delta\psi}_B. \tag{57}$$

Now the variation of the kinetic energy can be expressed in terms of intrinsic quantities only. To carry this out, a straightforward variation of the kinetic energy is performed

$$\int_0^l \delta K \,\mathrm{d}x_1 = \int_0^l \left[\delta V_B^{\mathrm{T}}\left(\frac{\partial K}{\partial V_B}\right)^{\mathrm{T}} + \delta\Omega_B^{\mathrm{T}}\left(\frac{\partial K}{\partial \Omega_B}\right)^{\mathrm{T}}\right]\mathrm{d}x_1. \tag{58}$$

Introducing sectional linear and angular momenta P_B and H_B that are conjugate to the generalized speeds

$$\begin{aligned} P_B &= \left(\frac{\partial K}{\partial V_B}\right)^{\mathrm{T}} = m(V_B - \tilde{\xi}_B\Omega_B) \\ H_B &= \left(\frac{\partial K}{\partial \Omega_B}\right)^{\mathrm{T}} = i_B\Omega + m\tilde{\xi}_B V_B \end{aligned} \tag{59}$$

one can write the variation of the kinetic energy density as

$$\begin{aligned} \delta K &= \delta V_B^{\mathrm{T}} P_B + \delta\Omega_B^{\mathrm{T}} H_B \\ &= (\dot{\overline{\delta q}}{}_B^{\mathrm{T}} - \overline{\delta q}{}_B^{\mathrm{T}}\tilde{\Omega}_B - \overline{\delta\psi}{}_B^{\mathrm{T}}\tilde{V}_B)P_B + (\dot{\overline{\delta\psi}}{}_B^{\mathrm{T}} - \overline{\delta\psi}{}_B^{\mathrm{T}}\tilde{\Omega}_B)H_B \end{aligned} \tag{60}$$

where the derived transpositional relations eqns (53) and (57) were used. As with the strain energy density, this final expression is independent of not only displacement and orientation variables, but of material properties as well.

Exact intrinsic equations

Known body forces and applied tractions over the surface of the beam can be shown (see Danielson and Hodges, 1988) to have virtual work equivalent to that of distributed applied forces and moments. The virtual work per unit length can be written as

$$\overline{\delta W} = \int_0^l (\overline{\delta q}_B^{\mathrm{T}} f_B + \overline{\delta \psi}_B^{\mathrm{T}} m_B)\, \mathrm{d}x_1. \tag{61}$$

With the virtual work, it is now possible to write down the exact intrinsic equations from eqns (24). First, in weak form

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_0^l \{(\dot{\overline{\delta q}}_B^{\mathrm{T}} - \overline{\delta q}_B^{\mathrm{T}}\tilde{\Omega}_B - \overline{\delta\psi}_B^{\mathrm{T}}\tilde{V}_B)P_B + (\dot{\overline{\delta\psi}}_B^{\mathrm{T}} - \overline{\delta\psi}_B^{\mathrm{T}}\tilde{\Omega}_B)H_B \\
&\quad -[(\overline{\delta q}'_B)^{\mathrm{T}} - \overline{\delta q}_B^{\mathrm{T}}\tilde{K}_B - \overline{\delta\psi}_B^{\mathrm{T}}(\tilde{e}_1+\tilde{\gamma})]F_B - [(\overline{\delta\psi}'_B)^{\mathrm{T}} - \overline{\delta\psi}_B^{\mathrm{T}}\tilde{K}_B]M_B + \overline{\delta q}_B^{\mathrm{T}} f_B + \overline{\delta\psi}_B^{\mathrm{T}} m_B\}\,\mathrm{d}x_1\,\mathrm{d}t \\
&\qquad = \int_0^l (\overline{\delta q}_B^{\mathrm{T}}\hat{P}_B + \overline{\delta\psi}_B^{\mathrm{T}}\hat{H}_B)|_{t_1}^{t_2}\,\mathrm{d}x_1 - \int_{t_1}^{t_2}(\overline{\delta q}_B^{\mathrm{T}}\hat{F}_B + \overline{\delta\psi}_B^{\mathrm{T}}\hat{M}_B)|_0^l\,\mathrm{d}t
\end{aligned} \tag{62}$$

where the quantities on the right-hand side with a $(\hat{\ })$ (hat) are the discrete boundary values of the force, moment, linear momentum and angular momentum. If there are discrete applied forces or moments at either end of the beam or discrete linear or angular impulses at either end of the time interval, these hatted quantities assume the values of those known quantities. If instead the displacements or rotations are known at the boundaries of either space or time, these hatted quantities become unknowns. When their values are found, values of internal forces, moments, linear momentum and angular momentum can be determined at the boundaries. In either case, the virtual displacements are unconstrained giving a fixed number of finite element equations and enough extra unknowns that either displacements or forces and either rotations or moments can be prescribed at the ends of the beam and either displacements or linear momenta and either rotations or angular momenta can be prescribed at the ends of the time interval. This approach is ideally suited for finite elements in that one can recover stresses or strains without any differentiation. Peters and Izadpanah (1988) demonstrate a similar method for a displacement approach in space and in time. A mixed formulation is given in the next section. Additional applications including mixed finite elements in space and time will be given in subsequent papers.

After integrating eqn (62) by parts, one obtains

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_0^l \{\overline{\delta q}_B^{\mathrm{T}}(F'_B + \tilde{K}_B F_B + f_B - \dot{P}_B - \tilde{\Omega}_B P_B) + \overline{\delta\psi}_B^{\mathrm{T}}[M'_B + \tilde{K}_B M_B + (\tilde{e}_1+\tilde{\gamma})F_B + m_B \\
&\quad - \dot{H}_B - \tilde{\Omega}_B H_B - \tilde{V}_B P_B]\}\,\mathrm{d}x_1\,\mathrm{d}t = \int_0^l [\overline{\delta q}_B^{\mathrm{T}}(\hat{P}_B - P_B) + \overline{\delta\psi}_B^{\mathrm{T}}(\hat{H}_B - H_B)]|_{t_1}^{t_2}\,\mathrm{d}x_1 \\
&\qquad - \int_{t_1}^{t_2}[\overline{\delta q}_B^{\mathrm{T}}(\hat{F}_B - F_B) + \overline{\delta\psi}_B^{\mathrm{T}}(\hat{M}_B - M_B)]|_0^l\,\mathrm{d}t
\end{aligned} \tag{63}$$

from which the Euler–Lagrange equations can be written as

$$\begin{aligned}
F'_B + \tilde{K}_B F_B + f_B &= \dot{P}_B + \tilde{\Omega}_B P_B \\
M'_B + \tilde{K}_B M_B + (\tilde{e}_1+\tilde{\gamma})F_B + m_B &= \dot{H}_B + \tilde{\Omega}_B H_B + \tilde{V}_B P_B.
\end{aligned} \tag{64}$$

The variational form in eqn (63) also leads to a consistent set of boundary conditions in which either force or moment can be specified or found at the ends of the beam.

Equations (64) are geometrically exact equations for the dynamics of a beam in a frame A whose inertial motion is arbitrary and known. These equations, when specialized to the static case, are identical to those of Reissner (1973). When specialized to the case of null inertial motion of A, they correspond to those of Simo and Vu-Quoc (1988). In other words, the present equations contain the given motion of the frame A explicitly, whereas in the work of Simo and Vu-Quoc (1988) this motion is implicit. If one desires to incorporate the (unknown) motion of a floating frame of reference, additional equations governing the motion of this frame would be required. This aspect is beyond the scope of the present work.

Here the measure numbers of virtual rotation $\overline{\delta\psi}_{Bi}$ for $i = 1, 2, 3$ are regarded as independent quantities, thus leading to eqn (64). If, instead, three appropriate rotational variables were used to express $\overline{\delta\psi}_{Bi}$, then variations of those variables could be regarded as independent. Since $\overline{\delta\psi}_B$ would be a linear combination of those variations, the resulting Euler–Lagrange equations would be a linear combination of the scalar equations in eqn (64). In other words, there is only one set of intrinsic equations; all other correct and variationally consistent sets of beam equations must be linear combinations of the correct intrinsic set of equations. The intrinsic equations have certain clearcut advantages; namely, their close resemblance and relationship to Euler's dynamical equations for a rigid body and their systematic form which enables one to write them in compact matrix notation.

MIXED VARIATIONAL FORMULATION

In the above derivation, spatial and temporal kinematical and constitutive relations have been used in order to finally obtain a set of intrinsic equations of motion for a beam. The final intrinsic equations are not really a stand-alone set of equations, however; since without the kinematical and constitutive relations they cannot be used to solve problems in general. In order to have a complete formulation, one can use the kinematical and constitutive relations to express the equations of motion in terms of some set of displacement and rotational variables. This is usually quite cumbersome, however, resulting in very complex equations. Hodges and Dowell (1974) contains a typical displacement formulation for a very restrictive set of assumptions; even so, the final equations of motion fill several pages.

The use of equations typical of these in a displacement finite element formulation is known to be cumbersome for at least two reasons. First, the equations are very long and complicated for general nonlinear behavior. Second, the orientation variables usually are exhibited in expressions involving transcendental functions. After carrying out the differentiations required to combine the equations, one can expand these functions in terms of polynomials of the unknowns and truncate the polynomials. This way, the integrals over individual finite elements can be carried out in closed form once and later recalled rather than recalculated. However, the truncation can result in unnecessary inaccuracies. Alternatively, one can numerically evaluate these integrals during each iteration when solving the nonlinear equations; while accurate, this latter approach can severely limit the computational efficiency.

An attractive alternative to the above is to combine all the necessary ingredients in a single variational formulation, leaving the generalized speeds, strains, forces, momenta and displacements as independent quantities. This can be done easily by adjoining the appropriate kinematical and constitutive relations to Hamilton's weak principle with Lagrange multipliers. Before this can be done, a set of rotational variables must be chosen. Since the displacement and rotational variables only appear in the kinematical relations, the choice of rotational variable does not impact very much of the formulation; it is, instead, localized to an equation for C and consistent equations for κ and Ω_B, as pointed out in Hodges (1987a).

For the purpose of illustration, Rodrigues parameters $\theta_i = 2e_i \tan(\alpha/2)$ are defined in terms of a rotation of magnitude α about a unit vector $\mathbf{e} = e_i\mathbf{b}_i$. In terms of a column matrix of the Rodrigues parameters

$$\theta = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} \tag{65}$$

the direction cosine matrix can easily be expressed as

$$C = \frac{\left(1 - \frac{\theta^T\theta}{4}\right)\Delta - \tilde{\theta} + \frac{\theta\theta^T}{2}}{1 + \frac{\theta^T\theta}{4}}. \tag{66}$$

The inverses of the kinematical relations are more convenient for obtaining a weak formulation, because they facilitate integration by parts to remove terms in which the unknowns are differentiated. The inverses of eqns (20) and (46) are, respectively,

$$u_b' = C^T(e_1 + \gamma) - e_1 - \tilde{k}_b u_b \tag{67}$$

and

$$\dot{u}_b = C^T V_B - v_b - \tilde{\omega}_b u_b. \tag{68}$$

Substitution of eqn (66) into eqn (23) leads to

$$\kappa = \left[\frac{\Delta - \frac{\tilde{\theta}}{2}}{1 + \frac{\theta^T\theta}{4}}\right] \theta' + Ck_b - k_b. \tag{69}$$

The inverse of this relationship is given by

$$\theta' = \left(\Delta + \frac{\tilde{\theta}}{2} + \frac{\theta\theta^T}{4}\right)(\kappa + k_b - Ck_b). \tag{70}$$

Similarly, substitution of eqn (66) into eqn (45) leads to

$$\Omega_B = \left[\frac{\Delta - \frac{\tilde{\theta}}{2}}{1 + \frac{\theta^T\theta}{4}}\right] \dot{\theta} + C\omega_b \tag{71}$$

and the inverse of this relationship is given by

$$\dot{\theta} = \left(\Delta + \frac{\tilde{\theta}}{2} + \frac{\theta\theta^T}{4}\right)(\Omega_B - C\omega_b). \tag{72}$$

After combining the variational form of the equilibrium equations, eqn (62), with the constitutive relations, eqns (27) and (59), and with the inverse kinematical relations, eqns (67), (68), (70) and (72), the Lagrange multipliers can be determined. After introducing

$$\begin{aligned}
\overline{\delta F} &= C^T \delta F_B \\
\overline{\delta M} &= \left[\frac{\Delta + \frac{\tilde{\theta}}{2}}{1 + \frac{\theta^T\theta}{4}}\right] \delta M_B \\
\overline{\delta P} &= C^T \delta P_B \\
\overline{\delta H} &= \left[\frac{\Delta + \frac{\tilde{\theta}}{2}}{1 + \frac{\theta^T\theta}{4}}\right] \delta H_B
\end{aligned} \tag{73}$$

one obtains from these operations

$$\int_{t_1}^{t_2}\int_0^l\Big\{[(\overline{\delta q}_B')^{\mathrm T}-\overline{\delta q}_B^{\mathrm T}\tilde K_B-\overline{\delta\psi}_B^{\mathrm T}(\tilde e_1+\tilde\gamma)]F_B+[(\overline{\delta\psi}_B')^{\mathrm T}-\overline{\delta\psi}_B^{\mathrm T}\tilde K_B]M_B$$

$$-(\dot{\overline{\delta q}}_B^{\mathrm T}-\overline{\delta q}_B^{\mathrm T}\tilde\Omega_B-\overline{\delta\psi}_B^{\mathrm T}\tilde V_B)P_B-(\dot{\overline{\delta\psi}}_B^{\mathrm T}-\overline{\delta\psi}_B^{\mathrm T}\tilde\Omega_B)H_B+\delta\gamma^{\mathrm T}\left[\left(\frac{\partial U}{\partial\gamma}\right)^{\mathrm T}-F_B\right]$$

$$+\delta\kappa^{\mathrm T}\left[\left(\frac{\partial U}{\partial\kappa}\right)^{\mathrm T}-M_B\right]-\delta V_B^{\mathrm T}[m(V_B-\tilde\xi_B\Omega_B)-P_B]-\delta\Omega_B^{\mathrm T}(i_B\Omega_B+m\tilde\xi_BV_B-H_B)$$

$$+\overline{\delta F}^{\mathrm T}[e_1+\tilde k_bu_b-C^{\mathrm T}(e_1+\gamma)]-(\overline{\delta F}')^{\mathrm T}u_b-\overline{\delta P}^{\mathrm T}(v_b+\tilde\omega_bu_b-C^{\mathrm T}V_B)+\dot{\overline{\delta P}}^{\mathrm T}u_b$$

$$+\overline{\delta M}^{\mathrm T}\left(\Delta+\frac{\tilde\theta}{2}+\frac{\theta\theta^{\mathrm T}}{4}\right)(Ck_b-k_b-\kappa)-(\overline{\delta M}')^{\mathrm T}\theta-\overline{\delta H}^{\mathrm T}\left(\Delta+\frac{\tilde\theta}{2}+\frac{\theta\theta^{\mathrm T}}{4}\right)(C\omega_b-\Omega_B)+\dot{\overline{\delta H}}^{\mathrm T}\theta$$

$$-\overline{\delta q}_B^{\mathrm T}f_B-\overline{\delta\psi}_B^{\mathrm T}m_B\Big\}\,\mathrm dx_1\,\mathrm dt=-\int_0^l(\overline{\delta q}_B^{\mathrm T}\hat P_B+\overline{\delta\psi}_B^{\mathrm T}\hat H_B-\overline{\delta P}^{\mathrm T}\hat u_b-\overline{\delta H}^{\mathrm T}\hat\theta)\big|_{t_1}^{t_2}\,\mathrm dx_1$$

$$+\int_{t_1}^{t_2}(\overline{\delta q}_B^{\mathrm T}\hat F_B+\overline{\delta\psi}_B^{\mathrm T}\hat M_B-\overline{\delta F}^{\mathrm T}\hat u_b-\overline{\delta M}^{\mathrm T}\hat\theta)\big|_0^l\,\mathrm dt. \tag{74}$$

In eqn (74) an integration by parts has been performed in order to remove the derivatives of all unknowns, and appropriate "hatted" terms have been added so that all boundary conditions are natural. This one equation is a complete representation of the dynamics of a moving beam. One should compare this with equations for a complete displacement formulation which require several pages to record even in a simplified form.

Equation (74) can be simplified for the static case and the results agree with equations found in Atilgan (1989) derived from the field equations and boundary conditions. After some integrations by parts, the equations agree with the static mixed formulation of Iura and Atluri (1989). The reason for the integrations by parts is that the goal of their work did not involve putting the expression in the weakest possible form as the present one does. Thus, the requirements for the shape functions in the present formulation are less restrictive. Indeed, since the unknown variables are never differentiated, approximate numerical quadrature can be avoided with the present method† everywhere except in the applied load terms, if the simplest possible shape functions are used. For illustrations of the power of this approach, see Atilgan (1989) and Hodges and Bless (1990).

INTRINSIC EQUATIONS WITH ZERO SHEAR DEFORMATION

In order to facilitate comparison with some earlier works, a separate version is now developed for the case of zero shear deformation. This can be accomplished in two ways: (1) from the partial differential equations of motion, one can set the two shear strain measures equal to zero, solve for the two shear forces from the bending moment equations, and substitute the result into the force equilibrium equations; (2) or one can apply the Euler–Bernoulli constraint to the expression for virtual rotation. Either way, the same final equations of motion are obtained; however, the variational procedure also leads directly to the correct boundary conditions.

Following the second way, one applies the Euler–Bernoulli constraint to the virtual rotation $\overline{\delta\psi}_B$. Consider the strain–displacement relation from eqn (20). For the Euler–Bernoulli constraint, let $\gamma_{12}=\gamma_{13}=0$. This leads to $1+\gamma_{11}=s'$ and thus that

$$\gamma=(s'-1)e_1 \tag{75}$$

† The author gratefully acknowledges the suggestion of Prof. Marco Borri for this methodology.

where s' is the derivative of the arc-length along $\boldsymbol{R}$ with respect to x_1. In light of this, one can solve eqn (20) for the first row of C obtaining

$$C^{\mathrm{T}}e_1 = \frac{e_1 + u_b' + \tilde{k}_b u_b}{s'}. \tag{76}$$

Premultiplication of $C^{\mathrm{T}}e_1$ by $e_1^{\mathrm{T}}C$ leads to an expression for s' so that

$$s'^2 = (e_1 + u_b' + \tilde{k}_b u_b)^{\mathrm{T}}(e_1 + u_b' + \tilde{k}_b u_b). \tag{77}$$

The six generalized strains have been reduced to four. Since $\gamma_{11} = s' - 1$ and $\gamma_{12} = \gamma_{13} = 0$, the moment strains κ and γ_{11} are independent. Although the six generalized speeds are not independent, there does not appear to be any two that should be eliminated in preference to the other four. However, the displacement field can be described by four displacement variables. These four could be three orientation variables and the extension (as in the Kirchhoff–Love theory) or, say, the measure numbers of displacement contained in u_b and a suitable measure of torsion (as in older helicopter blade theories without shear deformation). In the latter case, the matrix C is parameterized by four variables. As pointed out by Hodges (1985, 1987a) two of the three Rodrigues parameters can be eliminated. After some algebraic manipulations, using eqn (76) to eliminate θ_2 and θ_3, one can obtain

$$\theta = \theta_1 e_1 + \frac{1}{1 + C_{11}}[2\tilde{e}_1 + \theta_1(\Delta - e_1 e_1^{\mathrm{T}})]C^{\mathrm{T}}e_1. \tag{78}$$

Using this, one can express the six generalized speeds in terms of four independent quantities. This equation is of the form of several others to be presented below, and it will probably be helpful to the reader to point out what its structure entails. The first term shows that θ_1 is an independent quantity. The remaining terms determine θ_2 and θ_3 as functions of the four independent quantities u_b and θ_1.

Now we will proceed with the development of intrinsic equations for this special case in terms of s' and the curvature vector. Using eqn (39) and taking the variation of γ in this form, one obtains

$$\delta\gamma = \delta s' e_1 = \overline{\delta q}'_B + \tilde{K}_B \overline{\delta q}_B + s'\tilde{e}_1 \overline{\delta\psi}_B \tag{79}$$

or, solving for the virtual rotation,

$$\tilde{e}_1 \overline{\delta\psi}_B = \frac{1}{s'}(\delta s' e_1 - \overline{\delta q}'_B - \tilde{K}_B \overline{\delta q}_B). \tag{80}$$

The matrix $\tilde{e}_1$ is singular; however, it is possible to solve for two of the elements of $\overline{\delta\psi}_B$ and leave one as an independent quantity. Multiplying both sides of the equation by $-\tilde{e}_1$ and using the second and fifth of eqns (11) one obtains

$$-\tilde{e}_1\tilde{e}_1\overline{\delta\psi}_B = (\Delta - e_1 e_1^{\mathrm{T}})\overline{\delta\psi}_B = \frac{\tilde{e}_1}{s'}(\overline{\delta q}'_B + \tilde{K}_B \overline{\delta q}_B) \tag{81}$$

or

$$\overline{\delta\psi}_B = \overline{\delta\psi}_{B1} e_1 + \frac{\tilde{e}_1}{s'}(\overline{\delta q}'_B + \tilde{K}_B \overline{\delta q}_B). \tag{82}$$

From the structures of this equation, it can be seen that $\overline{\delta\psi}_{B1}$ remains in the equation as an independent quantity, serving as the virtual torsional rotation, while $\overline{\delta\psi}_{B2}$ and $\overline{\delta\psi}_{B3}$ disappear from it and are replaced by functions of three virtual displacements and their derivatives, as expected. This leads to the following form of Hamilton's principle:

$$\int_{t_1}^{t_2}\int_0^l \{\overline{\delta\psi}_{B1}(M'_{B1}+K_{B2}M_{B3}-K_{B3}M_{B2}+m_{B1}-\dot{H}_{B1}-\Omega_{B2}H_{B3}+\Omega_{B3}H_{B2}-V_{B2}P_{B3}+V_{B3}P_{B2})$$

$$+(\overline{\delta q}_B)^{\mathrm{T}}(F'_B+\tilde{K}_BF_B+f_B-\dot{P}_B-\tilde{\Omega}_BP_B)\}\,\mathrm{d}x_1\,\mathrm{d}t = \int_0^l [\overline{\delta q}_B^{\mathrm{T}}(\hat{P}_B-P_B)+\overline{\delta\psi}_B^{\mathrm{T}}(\hat{H}_B-H_B)]|_{t_1}^{t_2}\,\mathrm{d}x_1$$

$$-\int_{t_1}^{t_2}[\overline{\delta q}_B^{\mathrm{T}}(\hat{F}_B-F_B)+\overline{\delta\psi}_B^{\mathrm{T}}(\hat{M}_B-M_B)]|_0^l\,\mathrm{d}t \quad (83)$$

where now

$$F_B = F_{B1}e_1+\frac{\tilde{e}_1}{s'}(M'_B+\tilde{K}_BM_B-\dot{H}_B-\tilde{\Omega}_BH_B-\tilde{V}_BP_B). \quad (84)$$

A careful study of eqn (84) reveals that it is precisely what one would obtain from setting $e_1+\gamma = s'e_1$ in the second of eqns (64), premultiplying this equation by $-\tilde{e}_1$ and solving it for F_B. With this definition for the shear forces, the boundary conditions on the shear force are unchanged from the general formulation. Note, however, that the two shear forces are no longer available as partial derivatives of the strain energy with respect to the shear strains since the shear strains are no longer contained in the strain energy. The internal tension force is $F_{B1} = \partial U/\partial\gamma_{11}$ where $\gamma_{11} = s'-1$. The second and third elements of F_B involve other quantities (i.e. the shear forces are no longer independent internal generalized forces).

The intrinsic equations for the dynamics of an Euler–Bernoulli beam in a moving frame can now be written from the above. The torsion equation is a scalar equation given by

$$M'_{B1}+K_{B2}M_{B3}-K_{B3}M_{B2}+m_{B1} = \dot{H}_{B1}+\Omega_{B2}H_{B3}-\Omega_{B3}H_{B2}+V_{B2}P_{B3}-V_{B3}P_{B2} \quad (85)$$

whereas the three force equilibrium equations can still be written in compact notation as

$$F'_B+\tilde{K}_BF_B+f_B = \dot{P}_B+\tilde{\Omega}_BP_B \quad (86)$$

where F_B is given by eqn (84). Alternatively, one can regard the elements of $\overline{\delta q}_b$ as the virtual displacements and obtain

$$F'_b+\tilde{k}_bF_b+f_b = \dot{P}_b+\tilde{\omega}_bP_b \quad (87)$$

where $F_b = C^{\mathrm{T}}F_B$. Some investigators prefer the A frame basis thereby regarding elements of $\overline{\delta q}_A$ as the virtual displacements

$$F'_A+f_A = \dot{P}_A+\tilde{\omega}_AP_A \quad (88)$$

with $F_A = C^{AB}F_B$. Equation (85) and any one of eqns (86)–(88) and with F_B given by eqn (84) are the correct dynamical equations for deformation of an initially curved and twisted Euler–Bernoulli beam in a moving frame.

This approach also leads to very simple expressions for the curvature of an Euler–Bernoulli beam. For example, differentiating eqn (20) with respect to x_1, while noting that

$e_1+\gamma = s'e_1$, and solving for the curvature in a manner similar to the way virtual rotation was obtained for the Euler–Bernoulli case above, one can obtain

$$K_B - k_B = e_1(K_{B1} - k_{B1}) + \frac{\tilde{e}_1}{s'} C(u_b' + \tilde{k}_b u_b)'. \tag{89}$$

Thus, the torsion K_{B1} is independent of the bending curvatures. To compare with previous work, consider an initially straight beam for which the frame A is chosen so that $\mathbf{A}_1$ is along the undeformed beam axis. Now

$$K_B = e_1 K_{B1} + \frac{\tilde{e}_1 C^{BA} u_A''}{s'}. \tag{90}$$

It is now clear that if the torsion is represented by pretwist plus κ_1, one could choose κ_1 or its first integral to be the torsion variable. The first integral is a spatial quasi-coordinate and has been used by Hodges and Dowell (1974) and others. One problem with this approach is that C^{BA} must be expressed in terms of this quasi-coordinate, which leads to complicated expressions; see Alkire (1984). Additional discussion of torsion variables and the forms of κ and C for initially straight beams is given by Hodges *et al.* (1980) and Hodges (1987a). Equation (90) also reveals the possibility of neglecting the axial strain in the curvatures but retaining it elsewhere as done by Hodges (1985). Further reduction of the present equations for inextensionality (i.e. eliminating one of the displacement variables via the constraint $s' = 1$) is also possible, but will not be addressed in this paper.

These equations for dynamics of initially curved and twisted beams agree with equations obtained as an intermediate step and described as exact by Rosen and Friedmann (1977) for initially straight beams. However, Rosen and Friedmann did not express their equations in a compact matrix notation. Furthermore, they did not retain the exact form of the equations as their development progressed, preferring instead to obtain approximate equations via an ordering scheme. Their approximate equations, while correct to $O(\varepsilon^2)$, where ε is a book-keeping parameter determining the degree to which powers and products of unknowns are kept, are not correct to $O(\varepsilon^3)$ as pointed out by Hodges *et al.* (1988). Rosen's and Friedmann's final approximate equations cannot be put into compact matrix notation because they do not exhibit the appropriate patterns of symmetric and anti-symmetric operators in the present equations. These patterns are lost and the errors to $O(\varepsilon^3)$ are introduced in their equations primarily because they approximated the diagonal elements of C to be unity. Finally, their expressions for virtual rotation measure numbers are incorrect because of this same approximation of C having been made prior to their taking the variation of C to obtain the virtual rotation. Correct treatment of the virtual rotation for beams in terms of explicit displacement and rotational variables is found in Hodges *et al.* (1980) and Hodges (1985, 1987a).

The present variational formulation leads to a simple explanation of how one can introduce different representations for finite rotation (such as Rodrigues parameters, different orientation angles, etc.) and obtain equivalent but different final equations of motion. (For the fundamentals of finite rotation, see Kane *et al.* 1983; for a discussion related to beams see Hodges 1987a.) Both virtual displacement and virtual rotation measures can either be left as they are or expressed as a linear combination of the variations of a set of chosen coordinates. In the latter case, the final equations for one choice of variable turn out to be a linear combination of the intrinsic equations while for another choice of variable they turn out to be a different linear combination of the intrinsic equations. This is discussed further by Hodges *et al.* (1980). The intrinsic equations are essentially Newtonian in character and in general differ from those obtained in a specific variational formulation. Thus, the explanation given by Hodges and Dowell (1974) for the differences between their two (variational and Newtonian) formulations is correct. The present development, however, offers greater insight into the reasons for these differences.

While the equations in this last section are useful for numerical solution as well as for understanding the fundamentals of the theory, for mixed finite element analysis of beams without shear deformation it is no doubt simpler to proceed on the basis of eqn (74) with γ set equal to $\gamma_{11}e_1$. This way, the weak form of the general formulation can be exploited.

CONCLUSION

An intrinsic formulation for the dynamics of initially curved and twisted beams in a moving frame has been presented. Since the constitutive law is left in a generic form, the formulation implicitly includes the effects of in- and out-of-plane St Venant warping for beams with closed cross-sections and without constraints on the warping displacements. Atilgan and Hodges (1989) discuss in more detail the limitations of using only the six generalized strain variables used in the present analysis.

The equations are written in a compact matrix form without any approximations in the geometry of the deformed beam reference line or intrinsic cross-section frame. When specialized for the case of statics, they agree with a similar exact treatment by Reissner (1973). When specialized to a form in which all the frame motion is implicitly contained in the beam kinematical variables, they agree with the equations of Simo and Vu-Quoc (1988). When specialized to the case of zero shear deformation, they exhibit a simple form that is missing in previously published formulations even though the present equations are valid for initially curved and twisted beams.

Although the resulting equations are Newtonian in structure (closely resembling Euler's dynamical equations for a rigid body), the formulation is variational throughout, thus providing a link between Newtonian and energy-based methods. In particular, the present development provides substantial insight into the relationships among variational formulations in which different displacement and rotational variables are used as well as between these formulations and Newtonian ones.

Finally, a complete variational formulation of the mixed type is given for use in applications of the theory. The Euler–Lagrange equations are the kinematical, constitutive and equilibrium equations derived herein; and all the boundary conditions are natural ones. The choice of displacement and rotational variables is localized in a relatively small portion of the analysis, and none of the unknowns are ever differentiated with respect to space or time. Because of these features, it is possible to use very crude shape functions. Taking the field variables to be constant over each element in space and time, and allowing for discontinuities at the element boundaries, one can circumvent the use of numerical quadrature over the elements. Applications of this formulation to finite elements will be addressed in later papers.

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